

QUANTUM FORMALISM: BRIEF EPISTEMOLOGICAL CONSIDERATIONS.

HELEN LYNN AND MICHELE CAPONIGRO

ABSTRACT. We argue about a conceptual approach to quantum formalism. Starting from philosophical conjectures (Platonism, Idealism and Realism) as basic ontic elements (namely: math world, data world, and state of matter), we will analyze the quantum superposition principle. This analysis bring us to demonstrate that the basic assumptions affect in different ways: (a) the general problem of the information and computability about a system, (b) the nature of the math tool utilized and (c) the correspondent physical reality.

1. PLATONISM.

We are going to analyze the quantum superposition¹, while in the next section, starting from Platonic² view (where the math world is the ontic element), we will argue about the possible consequences of the analysis on : (i)the notion of information and computability of the system (ii)the math tool and (iii)the correspondent physical reality.

1.1. Basic: Quantum superposition. As we know, the essential difference of the quantum mechanical concept of reality from usual classical reality is that in quantum mechanics the properties of material systems, as they are observed in a measurement, may not exist **before** the observation (measurement process). If for example the measurement shows that a particle is located in one of two points A_1 ,

¹Quantum superposition is fundamental feature of QM. For instance, the following two questions can be combined into a single question, “Why the representation by Hilbert space?”

- (1) Why do we have to represent quantum processes and states by *complex-valued* mathematical objects?
- (2) Why the superposition principle? — That is, why do we represent quantum states and processes by objects that add up linearly?

Heisenberg, for example, wrote about the superposition (or interference of potentia): *is the most characteristic phenomenon of quantum theory, is destroyed by the partly indefinable and irreversible interactions of the system with the measuring apparatus and the rest of the world.* Certain interpretations of quantum mechanics do so by treating the superposition of co-existent potentia in the wave function as a superposition of co-existent actualities (Everett in his Relative State interpretation). In few words, the benefit of using Hilbert spaces in quantum mechanics is that these spaces are capable of representing, in a mathematically useful way, **potentia as well as actualities and their relationship in a given system**. The representation is based on two simple principles: (i) every physical system can be represented by a unique Hilbert space, and the state S of a given physical system can be represented by a single vector of unit length in the systems Hilbert space. (ii) In a measurement interaction involving the system, there is a one-to-one correspondence between the number of probability-valuated outcome system states and the number of dimensions comprised by the Hilbert space. The fact that these are mutually orthogonal vectors is representative of the mutual exclusivity of the states. It is crucial to note that ontologically, the desideratum of **the mutual exclusivity of probable alternative outcome states is a presupposed principle a first principle of quantum mechanics in that it has not, as yet, been deduced from some more fundamental physical principle**. Alternatively, many have argued that the mutual exclusivity of outcome states is merely an epistemic phenomenon and thus ontologically insignificant. For instance, it is possible in a **less idealized Hilbert space**, simply have a group of mutually orthogonal vectors into subspaces, where any vector belonging to the subspace is orthogonal to any vector in subspace. This would imply, however, that there exist vectors of unit length (i.e., vectors that represent real states of a physical system) that belong neither, there are, in other words, at least in the early phases of quantum mechanical state evolution, real physical states wherein the actuality of a given potential fact is objectively indefinite.

²Platonism about mathematical entities is the dominant realist tradition. Platonist believe that mathematical entities have an existence independent of human minds. These entities inhabit a special world, the Platonic world. Platonism thus believes not only in the independent existence of mathematical objects and relations but also believes that the “reality” of that world explains the universal nature of mathematical truth. However, Platonism, runs into serious problems when confronted with the **applicability** of mathematics. In this case, the basic problem is to understand how these Platonic entities, which do not have spatial or temporal characteristics, can get in touch with our physical world, which is defined by spatio-temporal extension. In other words, how do we as humans access these Platonic objects? And how do these objects link up with our real world?

A_2 , this particle may be located neither in A_1 , nor in A_2 **before the measurement**. Moreover are not considerate the finite spatial and temporal differentiation of the physical world, which implies that no physical system can, during a finite time span T , be in an infinite number of states such that each state is distinct from (orthogonal to) its immediate predecessor. To see how this plays out in real physics, consider the quantum superposition:

$$\psi = \sum_i c_i \varphi_i \quad (1)$$

in case of simple quantum superposition of two eigenstates φ_1, φ_2 , we find the following state of the particle before the measurement: $\psi = c_1\varphi_1 + c_2\varphi_2$, this superposition of states is localized correspondingly in A_1 and A_2 . According to von Neumann³ reduction postulate, after the measurement distinguishing between

³In von Neumann's formalism, the central role is played by a Hilbert space (**contrary to Heisenberg and Pauli who were intellectually close to Bohr, von Neumann represented another line of ideas that originated with the mathematician Hilbert. The latter had another conception of the nature of a physical theory: the axioms**), and wave functions were just elements of the convenient representation of such a space. It is true, generations of experimental physicists, chemists, etc. were using wave functions and Schroedinger's equations without ever thinking about Hilbert spaces, but it is a matter of practical applications in which the choice of concepts and their understanding are secondary issues. Von Neumann provided such a consistent formalism in terms which did not use **the concept of a wave function as fundamental**. Actually, he even assumed that quantum state is not described by a vector in the Hilbert space, but by a one dimensional subspace or equivalently by a projector on such a subspace. So, **a wave function disappeared from the view**. It was long way from de Broglie who believed that the wave functions are actual physical waves. After von Neumann published his Foundations several other different formalisms have been introduced, such as algebraic formalism based on C^* algebras (started 1934 by Jordan, von Neumann, and Wigner, developed later in different variations by Segal, Haag and Koestler, Gelfand and Naimark), convex state space approach (initiated by Stone, and von Neumann and Morgenstern, and developed by Mielnik, Ludwig, Davies and Lewis), quantum logic approach (initiated 1936 by Birkhoff and von Neumann, developed by Jauch and Piron and many others). It is amazing that each time the name of von Neumann appears among initiators of the very different approaches. Von Neumann in effect proposed the following quantum-theoretical reinterpretations:

- Phase space $M \longrightarrow$ Hilbert space H ;
- Classical observable (i.e. real-valued measurable function on M) \longrightarrow self-adjoint operator on H ;
- Pure state (seen as point in M) \longrightarrow unit vector (actually ray) in H ;
- Mixed state (i.e. probability measure on M) \longrightarrow density matrix on H ;
- Measurable subset of $M \longrightarrow$ closed linear subspace of H ;
- Set complement \longrightarrow orthogonal complement;
- Union of subsets \longrightarrow closed linear span of subspaces;
- Intersection of subsets \longrightarrow intersection of subspaces;
- Yes-no question (i.e. characteristic function on M) \longrightarrow projection operator;

After the formulation of QM Physicists was interested to search new formalisms, the reason was linked two main motivations: (i) first, there was no fully general and consistent relativistic QM formalism. The second motivation was to get rid of the elements of the theory which are clearly redundant. As above mentioned the wave functions has been removed from the picture because they involved many elements which did not have any physical or philosophical interpretation. The formalism presented in von Neumann's Foundations still had a lot of redundancy. By the time of Foundations it was clear that any formalism has to describe the two notions, of the state(s) of a system and of the observable(s). It is interesting that before QM was born the state of the system has been almost completely neglected. It was considered obvious that the state of the system is directly given by the values of observables. In classical mechanics, when we have enough functions on the state space (phase space), the state can be identified uniquely as the unique element of

these two alternatives, the system having been previously in the state ψ goes over into one of the states ψ_1 and ψ_2 , with the corresponding probabilities $|c_1|^2$ and $|c_2|^2$. This postulate corresponds to what is observed in real measurements, so the reduction postulate is accepted as the basis for the quantum-mechanical calculations. However, as we know, it contradicts to the linearity of quantum mechanics when the process of measurement is considered as an interaction of two systems (the measured system and the measuring device).

1.2. Platonism and Complex numbers. In this view, the ontic element is the math world. Speaking of quantum superposition the amplitudes c_1 and c_2 are complex numbers⁴ which, in general, demand an **infinite amount of information to specify them precisely**. The state does not compute in the (resource-limited)

the intersection of inverse images of the values of functions representing observables. In QM the separation of the roles of the concept of a state and of observables has become fundamental. In classical mechanics (CM) every observable has some unique and clearly determined value in each state of the system (the state was identified as a point in the phase space, and the observable was a function on this space). Now, in quantum mechanics (QM) it turned out that in some states the system may have specific value of an observable, in some states not. In conclusion, the postulate on the completeness of QM is not so innocent, it is not just a philosophic subject but has important implications.

⁴There are detailed works with the objective to avoid the presence of complex numbers in QM, for instance the utilization of Wigner quasi-distribution. The Wigner quasi-probability distribution was introduced by Wigner in 1932 to study quantum corrections to classical statistical mechanics. The goal was to replace the wavefunction that appears in Schrödingers equation with a probability distribution in phase space. It was independently derived by Hermann Weyl in 1931 as the symbol of the density matrix in representation theory in mathematics. It was once again derived by J. Ville in 1948 as a quadratic (in signal) representation of the local time-frequency energy of a signal. It is also known as the "Wigner function," "Wigner-Weyl transformation" or the "Wigner-Ville distribution". It has applications in statistical mechanics, a classical particle has a definite position and momentum and hence, is represented by a point in phase space. When one has a collection (ensemble) of particles, the probability of finding a particle at a certain position in phase space is given by a probability distribution. This is not true for a quantum particle due to the uncertainty principle. Instead, one can create a quasi-probability distribution, which necessarily does not satisfy all the properties of a normal probability distribution. For instance, the Wigner distribution can go negative for states which have no classical model (and hence, it can be used to identify non-classical states). For instance, the solution and visualization of problems in one-dimensional quantum mechanics focuses most often on calculations of the position-space wavefunction, $\psi(x, t)$. More occasionally, problems may be solved using, or transformed into, the momentum-space counterpart, $\phi(p, t)$, using the standard Fourier transforms,

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \phi(p, t) e^{+ipx/\hbar} dp \quad (2)$$

and

$$\phi(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x, t) e^{-ipx/\hbar} dx. \quad (3)$$

Connections between the classical and quantum descriptions of model systems can then be made in a variety of ways. For example, the quantum mechanical expectation values, $\langle x \rangle_t$ and $\langle p \rangle_t$ can be compared to their classical analogs, $x(t)$ and $p(t) = mv(t)$, via Ehrenfest's principle, e.g., $\langle p \rangle_t = md\langle x \rangle_t/dt$. Quantum mechanical probability densities, $P_{QM}^{(n)}(x) = |\psi_n(x)|^2$ and $P_{QM}^{(n)}(p) = |\phi_n(p)|^2$, can be related to classical probability distributions. The visualization of solutions to problems in classical mechanics through a phase-space description, i.e., parametric plots of $p(t)$ versus $x(t)$, is often helpful. It is natural to wonder if a quantum mechanical analog of a phase-space probability distribution, a joint $P(x, p)$ probability density, is a useful construct, despite the obvious problems raised by the Heisenberg uncertainty principle and its connection between x and p .

universe; it computes in the (infinitely resourced) Platonic realm, where they can be subjected to infinitely precise idealized mathematical operations such as unitary evolution. **In this framework, mathematical equations describe not merely some limited aspects of the physical world, but all aspects of it.** It means that there is some mathematical structure that is what mathematicians call isomorphic, and hence equivalent, to our physical world, with each physical entity having a unique counterpart in the mathematical structure⁵ and vice versa. We are asking if this mathematical structure is necessary and useful? In other words, the "infinite" structure give us more information about a system respect a finite structure. It is a correct statement? We are going to analyze the second assumption.

Wigner, as said before, was one of the first to address this issue and introduced a quasi- or pseudo-probability distribution corresponding to a general quantum state, $\psi(x, t)$, defined by

$$P_W(x, p; t) \equiv \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} \psi^*(x + y, t) \psi(x - y, t) e^{2ipy/\hbar} dy. \quad (4)$$

$P(x, p)$ has the important properties it is real. As we seen, complex numbers appear in the Hilbert space formulation of quantum mechanics, but not in the formulation in phase space. It is possible to conclude that the use of complex numbers in quantum mechanics can be regarded as a computational device to simplify calculations, as in all other applications of mathematics to physical phenomena. The essential place of complex numbers in quantum mechanics as usually formulated is evident in Schrödinger's time-dependent wave equation. The complex numbers are introduced only as a computational tool. Not only are the 'observables' of these models real but, invariably, so also are their defining equations. The phase-space formulation is purely real, there are no complex numbers to be seen in the defining equations. Despite this, it has been stated, that the phase-space formulation is equivalent to the more familiar formulation in terms of hermitian operators acting on a complex Hilbert space. Quantum symmetries and complex superpositions of pure, orthogonal quantum states, can both be described in the phase space formulation of quantum mechanics, without the use of complex numbers. The description of symmetries is much simpler in phase space, but the **description of superpositions is much more complicated**. Finally, the phase space formulation does indeed appear capable of reproducing all aspects of quantum mechanics. In the case of the superposition of quantum states however, this is only be achieved at the cost of much greater complication. If we wish to think of the phase space formulation as the more fundamental, arising directly from a deformation of classical mechanics in phase space, we can think of the formulation of quantum mechanics in Hilbert space, and the associated introduction of complex numbers, as a computational device to make calculations easier. From this point of view, the appearance of complex numbers in quantum mechanics is on a similar footing to their appearance in other applications of mathematics to natural phenomena.

⁵In general, Hilbert space contains infinite dimensions, but these are not geometric, rather, each dimension represents a state of possible existence for a quantum system, for instance, an electron **unmeasured** is a complicated pattern in an infinite-dimensional Hilbert space, the measure of its momentum in a given direction in space give us an infinite number of possible momenta. Each definite observable eigenvalue of momentum, is an independent axis, or dimension, or "choice", with a momentum eigenfunction. The infinite set of all such plane waves, of all possible wavelengths, in all possible directions in physical space, forms a momentum frame of reference for the quantum Hilbert space, which in this case is **infinitely dimensional**, there is a nondenumerable or noncountable infinity of dimensions to the quantum Hilbert space for the motion of this single particle. In conclusion the platonic Hilbert realm is characterized by nondenumerable state vectors (wavefunctions).

2. IDEALISM.

In this second assumption, the ontic element is: all possible **data of the system**. Now, we are obliged to assume that the universe computes in the (same) universe, and there **is not an infinite source** of free information in a Platonic realm at the disposal of Nature. The bound on universe applies to all forms of information, including such numbers as c_1 and c_2 , and to the dynamical evolution of the state vector ψ , and is not merely a bound on the number of degrees of freedom in the universe (or on the dimensionality of Hilbert space). We are interested to know **who** is getting all these possible data. Until now, we have not not speak about any observer or device. Is not necessary to introduce an observer[1]. We do not force to introduce any split between "objects". In the realm, that we call "Idealism", we accept a **finite information and computability**. An important difference between Platonism is that, here, we believe that all these denumerable data belong to the "someone", this object is not necessary to be an human observer. According this view, the world is not completely mathematical, it means that mathematical equations describe merely some limited aspects of the possible **finite "data" world**. In this case the finite data world could to fix rules in that of infinite mathematical world. **While the mathematical world seem an external and internal Idealism , the data world seem an idealism concentrate to the "subject"**.

3. REALISM.

We are going to analyze third case: the Realism⁶. We leaved the Platonic realm. The complex numbers "*collapse*" to real numbers (c_1 and c_2). The state of matter is the ontic element. We have a finite information and computability (like second case), here, we cannot speak about any Complex Spaces. The math tools is different. The same equation (e.g.superposition) change his nature (now the problems are linked to real numbers, integers). Moreover we have to consider a finite spatial and temporal differentiation of the physical world, which implies that no physical system can, during a finite time span T , be in an infinite number of states such that each state is distinct from (orthogonal to) its immediate predecessor. In this realistic framework, we conclude with the following citation (Pearle[2]): *There is a big difference between a conditional statement and an absolute statement: "if you win the lottery then you will get ten million dollars" can't compare with "you have won the lottery and you get ten million dollars."* *The statements of Standard QM are conditional.Faced with the statevector $c_1|a_1\rangle + c_2|a_2\rangle$, Standard QM says "if this is the description of a completed measurement then the physical state is $|a_1\rangle$ or $|a_2\rangle$."* *But actually, what the if is conditioned upon, what the words "a completed measurement" mean.....Qm fails to predict a physical phenomenon, namely that an event does—or does not—occur. Phenomenological models are introduced into physics to describe phenomena that present theory fails to adequately treat. Collapse models are phenomenological models. Their statements are absolute.Faced with the statevector $c_1|a_1\rangle + c_2|a_2\rangle$, the collapse*

⁶The notion of classical world includes mainly two ingredients: (i) realism (ii) determinism. By realism we means that any quantity that can be measured is well defined even if we do not measure it in practice. By determinism we mean that the result of a measurement is determined in a definite way by the state of the system and by the measurement setup. Quantum mechanics is not classical in both respects. As we know, QM requires four postulates: Two postulates define the notion of quantum state, while the other two postulates, in analogy with classical mechanics, are about the laws that govern the evolution of quantum mechanical systems.

model says that represents the physical state. If you wait a bit, it may happen that the statevector is unchanged, and that's that. Or, it may occur that the statevector rapidly evolves to $|a_1\rangle$ or $|a_2\rangle$, and again that's that. By "that's that" is meant, in all cases, that the statevector represents the physical state: the model tells you whether or not an event occurred. A summary of the positions regarding c_1 and c_2 :

Assumptions:	c_1 and c_2	$\psi = c_1\varphi_1 + c_2\varphi_2$
Platonism	infinite information (comput.)	<i>all infinite and idealized reality is everywhere</i>
Idealism	finite information (comput.)	<i>reality is a random data world</i>
Realism	finite inform. constrained, real	<i>reality is the state of matter</i>

4. CONCLUSION.

We conclude with the following table and citation:

Summary of the paper: Information, Computability, Physical world and Basic Assumptions

- Platonic view: **infinite ad idealized information**. All theoretical principles are here. The reality with all his infinite laws is everywhere.
- Idealistic view: **finite information**. A finite information without constraints. In this view, previous infinite information and computability do not give us more **data** world. According this view, we are not sure that reality is everywhere, because we get all finite data world by "subject". **An infinite inside/outside physical reality become probably a finite inside.**
- Realistic view: We meet the matter(ontic element), not only with finite information and computation but with constraints, here the math tools is different. The same equation (e.g. superposition) change his nature (problem of real numbers, integers). We have to consider, moreover, a finite spatial and temporal differentiation of the physical world, which implies that no physical system can, during a finite time span T , be in an infinite number of states such that each state is distinct from (orthogonal to) its immediate predecessor.

(Nikolić[3]): *Textbooks on QM usually emphasize the pragmatic technical aspects, while the discussions of the conceptual issues are usually avoided or reduced to simple authoritative claims without a detailed discussion. This causes a common (but wrong!) impression among physicists that all conceptual problems of QM are already solved or that the unsolved problems are not really physical (but rather "philosophical").*

REFERENCES

- [1] Rovelli C. Relational Quantum Mechanics, International Journal of Theoretical Physics, 35, 1637 (1996)
- [2] Pearle P. "Open Systems and Measurement in Relativistic Quantum Theory," F. Petruccione and H. P. Breuer eds. (Springer Verlag, 1999)
- [3] Nikolić H. Quantum Mechanics: Myths and facts: quant-ph069163 (Sept. 2006)

QUANTUM PHILOSOPHY THEORIES, WWW.QPT.ORG.UK

E-mail address: helen.lynn@qpt.org.uk

PHYSICS DEPARTMENT, UNIVERSITY OF CAMERINO, I-62032 CAMERINO, ITALY

E-mail address: michele.caponigro@unicam.it